Assignment 2.

This homework is due *Tuesday* 10/5/2010.

There are total 72 points in this assignment. 60 points is considered 100%. If you go over 60 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) (Exercises 3.1.1b, d, 3b, d) Write first five terms of the following sequences:
  (a) [1pt] $x_n = (-1)^n/n$,
  (b) [1pt] $x_n = \frac{1}{n^2+1}$,
  (c) [1pt] $x_1 = 2, x_{n+1} = \frac{1}{2}(x_n + 2/x_n)$,
  (d) [1pt] $x_1 = 3, x_2 = 5, x_{n+2} = x_n + x_{n+1}$.

(2) (a) [3pt] (Theorem 3.2.3) Let $X = (x_n)$ and $Y = (y_n)$ be sequences in $\mathbb{R}$ converging to x and y, respectively. Prove that $X - Y$ converges to $x - y$.
  (b) [3pt] (Exercise 3.2.3) Show that if X and Y are sequences in $\mathbb{R}$ such that X and $X + Y$ converge, then Y converges.
  (c) [3pt] (Exercise 3.2.2b) Give an example of two sequences X, Y in $\mathbb{R}$ such that XY converges, while X and Y do not.

(3) In each case below, find a number $K \in \mathbb{N}$ such that the corresponding inequality holds for all $n > K$. Give a *specific number* as your answer, for example $K = 1000$, or $K = 2 \cdot 10^7$, or $K = 139$, etc. (Not necessarily the smallest possible.)

  You can use a calculator if you want to. However, 1) this problem can be done without using a calculator, 2) even if you do use one, your answers still should be justified regardless of that.
  (a) [2pt] $\left|\frac{100-n}{n} - (-1)\right| < 0.054352$,
  (b) [2pt] $\left|\frac{200n+1000}{n^2-1001}\right| < 0.1$,
  (c) [2pt] $\left|\frac{0.2n^2-200n+1000}{0.1n^2-1001} - 2\right| < 0.1$,
  (d) [2pt] $\left|1/3^n - 1/n^2 + 100/n^5\right| < 0.01$,
  (e) [2pt] $\left|\frac{\cos(863n)}{\log n}\right| < 0.032432$,
  (f) [4pt] (See example 3.1.11(c)) $|\sqrt[n]{n} - 1| < 0.1$,
  (g) [3pt] $\left|\left(\sqrt{n^2+n+1} - n\right) - \frac{1}{2}\right| < 0.005$.

(4) Reminder. Recall definition of a sequence in $\mathbb{R}$ converging to an $x \in \mathbb{R}$: Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ converges to x if **for any** $\varepsilon > 0$, **there is** a $K \in \mathbb{N}$ such that **for any** $n > K$, the following inequality holds: $|x - x_n| < \varepsilon$.

Below you can find (erroneous!) "definitions" of a sequence converging to x. In each case describe, exactly which sequences are "converging to x" according to that "definition".

(a) [4pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if for any $\varepsilon > 0$, **for any** $K \in \mathbb{N}$, for any $n > K$, the following inequality holds: $|x - x_n| < \varepsilon$.
(*If you are confused at this point, think of the problem this way: suppose for some sequence* (x_n) *and a number* $x \in \mathbb{R}$ *you know that statement (a) is true. What can you say about about* (x_n)*?*)

(b) [4pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if **there is a** $K \in \mathbb{N}$ **such that for any** $\varepsilon > 0$, for any $n > K$, the following inequality holds: $|x - x_n| < \varepsilon$.

(c) [4pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ converges to x if **there is** an $\varepsilon > 0$ such that there is a $K \in \mathbb{N}$ such that for any $n > K$, the following inequality holds: $|x - x_n| < \varepsilon$.

(d) [6pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if for any $\varepsilon > 0$, there is a K such that **there is** $n > K \in \mathbb{N}$, the following inequality holds: $|x - x_n| < \varepsilon$.

(5) Reminder. Recall that a sequence $X = (x_n)$ in $\mathbb{R}$ **does not** converge to $x \in \mathbb{R}$ if there is an $\varepsilon_0 > 0$ such that for any $K \in \mathbb{N}$ there is $n_0 > K$ such that following inequality holds: $|x - x_n| \geq \varepsilon_0$.

In each case below find a *real number* ε_0 that demonstrates that (x_n) does not converge to x.

(a) [2pt] $x_n = 1 + 0.1 \cdot (-1)^{n+1}$, $x = 1$,

(b) [2pt] $x_n = 1/n$, $x = 1/17$,

(c) [2pt] $x_n = (-1)^n n^2$, $x = 0$.

(6) (Exercise 3.1.8) Let (x_n) be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$.

(a) [4pt] Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$.

(b) [3pt] Prove that if (x_n) converges to x then $(|x_n|)$ converges to $|x|$.

(c) [3pt] Give an example to show that the convergence of $(|x_n|)$ does not imply the convergence of (x_n).

(7) [4pt] (Exercise 3.2.7) If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 (Arithmetic properties of limit) *cannot* be used.

(8) [4pt] (Exercise 3.2.11) If $0 < a < b$, determine $\lim\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$.